Image Transformation

$$O : \mathcal{I} \to \mathcal{I}, \quad \mathcal{I} : \text{Image Space}.$$

If the operation O is linear,

then we can define the Point Spread Function:

$$h^{\alpha,\beta}(x,y) = \left[O\left(\begin{array}{c} \\ \\ x\text{th row} \\ \\ \end{array} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \right) \right]_{\alpha,\beta}$$

$$\begin{array}{c} \uparrow \\ y\text{th col} \end{array}$$

Meaning: How does the pixel (x,y) influence pixel (α,β) in the Transformation

Example

- $O_1(f) := A + f$, A non-zero matrix, is not linear, as

$$O_1(f+g) = A + f + g = (A+f) + (A+g) - A$$
$$= O_1(f) + O_1(g) - A \neq O_1(f) + O_1(g) \text{ as } A \neq 0$$

- $O_2(f) = cf$, $c \in \mathbb{R}$, is linear,

$$\text{as } O_2(af + bg) = caf + cbg = a(cf) + b(cg)$$
$$= a\,O_2(f) + b\,O_2(g).$$

- $O_3(f) = f^T f$ is not linear,

$$\text{as } O_3(2f) = 4 f^T f = 4\,O_3(f)$$

Example

$$\mathcal{O}(f) := Af \quad , \quad f \in \mathcal{I}$$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

First, $\mathcal{O}$ is linear:

$\forall f, g \in \mathcal{I}, a, b \in \mathbb{R}$,

$$\begin{aligned} \mathcal{O}(af + bg) &= A(af + bg) \\ &= aAf + bAg \\ &= a\mathcal{O}(f) + b\mathcal{O}(g). \end{aligned}$$

PSF:

$$\begin{aligned} h^{1,1}(1,1) &= \left(A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)_{1,1} \\ &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{bmatrix} \right)_{1,1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} h^{1,1}(2,3) &= \left(A \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)_{1,1} \\ &= \left(\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 8 \end{bmatrix} \right)_{1,1} \\ &= 0 \end{aligned}$$

Convolution (Discrete)

$$f, k \in \underline{I}$$

Periodically extend f :

$$f(x,y) = f(x+N, y+N) = f(x+2N, y+N)$$
$$= f(x+N, y+2N) = f(x-N, y-N)$$
$$= \dots$$

Convolution $k * f$ is an Image Transformation given by :

$$k * f(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} k(x,y)\, f(\alpha - x, \beta - y)$$

e.g.

$$k * f(1,1) = \sum_{x=1}^{3} \sum_{y=1}^{3} k(x,y)\, f(1-x, 1-y)$$

$$= k(1,1) f(0,0) + k(1,2) f(0,-1) + k(1,3) f(0,-2)$$
$$+ k(2,1) f(-1,0) + k(2,2) f(-1,-1) + k(2,3) f(-1,-2)$$
$$+ k(3,1) f(-2,0) + k(3,2) f(-2,-1) + k(3,3) f(-2,-2)$$

Rearrange

$$= k(3,3) f(-2,-2) + k(3,2) f(-2,-1) + k(3,1) f(-2,0)$$
$$+ k(2,3) f(-1,-2) + k(2,2) f(-1,-1) + k(2,1) f(-1,0)$$
$$+ k(1,3) f(0,-2) + k(1,2) f(0,-1) + k(1,1) f(0,0)$$

$$\begin{bmatrix} k(1,1) & k(1,2) & k(1,3) \\ k(2,1) & k(2,2) & k(2,3) \\ k(3,1) & k(3,2) & k(3,3) \end{bmatrix}$$

rotate by 180°

$$\Big\downarrow$$

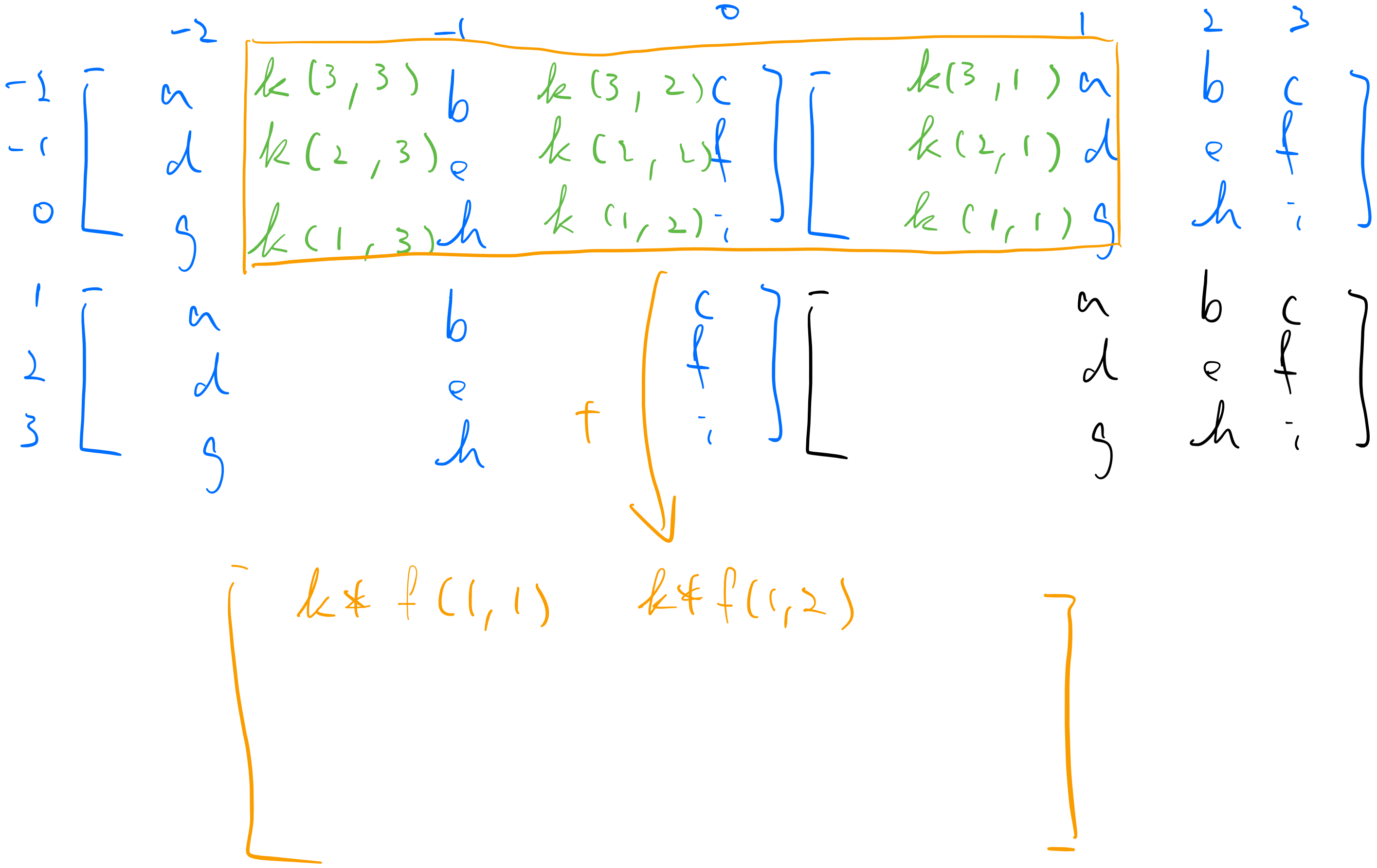
$$\begin{bmatrix} k(3,3) & k(3,2) & k(3,1) \\ k(2,3) & k(2,2) & k(2,1) \\ k(1,3) & k(1,2) & k(1,1) \end{bmatrix}$$

$$\begin{array}{c|cccccc} & -2 & -1 & 0 & 1 & 2 & 3 \\ \hline -2 & k(3,3)\,a & k(3,2)\,b & k(3,1)\,c & a & b & c \\ -1 & k(2,3)\,d & k(2,2)\,e & k(2,1)\,f & d & e & f \\ 0 & k(1,3)\,g & k(1,2)\,h & k(1,1)\,i & g & h & i \\ 1 & a & b & c & a & b & c \\ 2 & d & e & f & d & e & f \\ 3 & g & h & i & g & h & i \end{array}$$

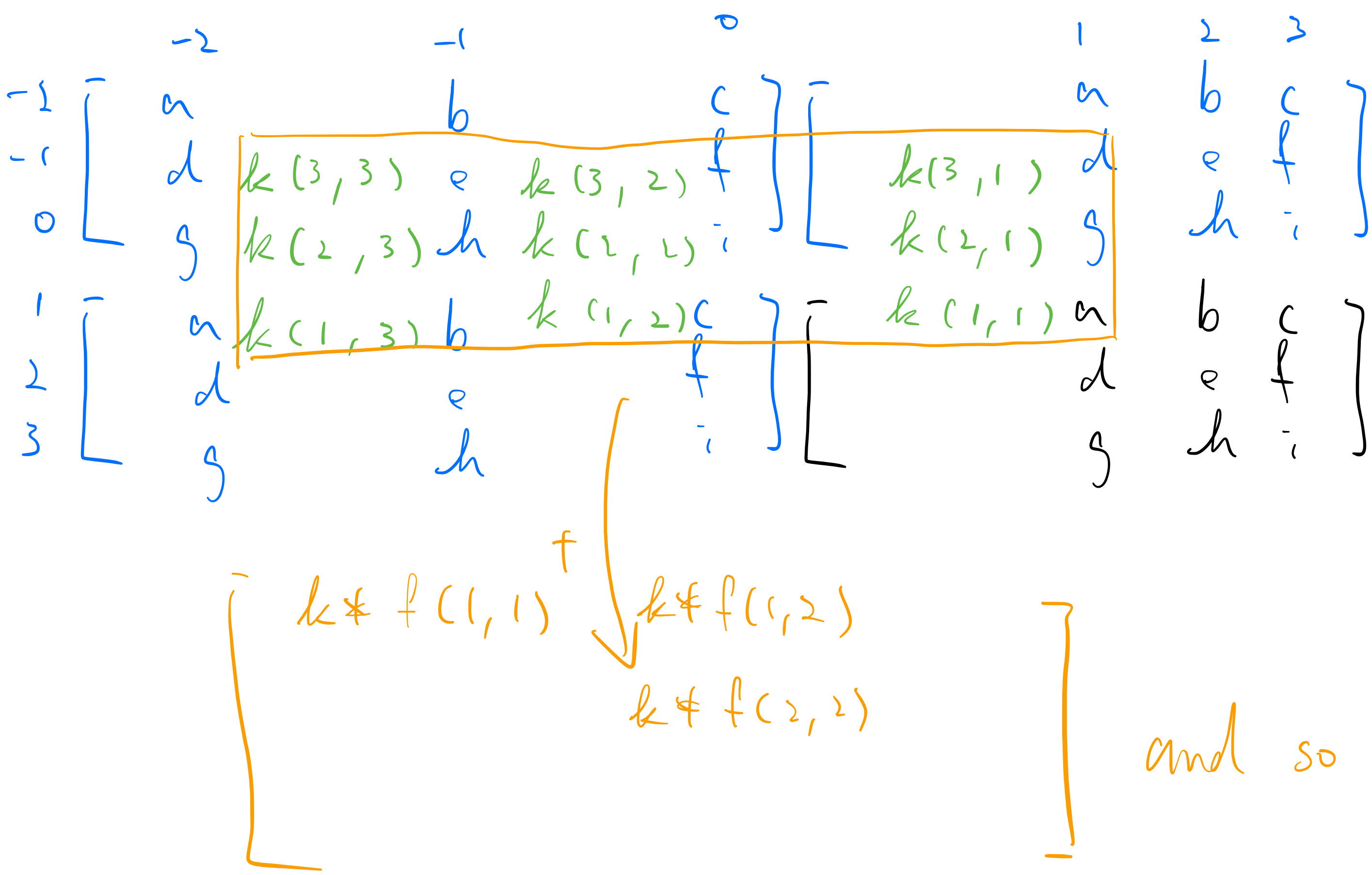
f

$$\Big\downarrow$$

$$\begin{bmatrix} k * f(1,1) & & \\ & & \\ & & \end{bmatrix}$$



-2 -1 0 1 2 3
k(3,3) k(3,2) k(3,1)
k(2,3) k(2,2) k(2,1)
k(1,3) k(1,2) k(1,1)
a b c
d e f
g h i
f
k*f(1,1) k*f(1,2)



-2 -1 0 1 2 3
k(3,3) k(3,2) k(3,1)
k(2,3) k(2,2) k(2,1)
k(1,3) k(1,2) k(1,1)
a b c
d e f
g h i
f
k*f(1,1) k*f(1,2)
k*f(2,2)

and so on

Convolution vs Cross Correlation

Convolution:

$$k * f(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} k(x,y)\, f(\alpha - x, \beta - y)$$

Cross Correlation:

$$k \star f(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} k(x,y)\, f(\alpha + x, \beta + y)$$

Convolution:

$$k: \begin{bmatrix} k(1,1) & \cdots & k(1,N) \\ \vdots & & \vdots \\ k(N,1) & \cdots & k(N,N) \end{bmatrix}$$

$\downarrow$ rotate by 180°

$$\begin{bmatrix} k(N,N) & \cdots & k(N,1) \\ \vdots & & \vdots \\ k(1,N) & \cdots & k(1,1) \end{bmatrix}$$

$\downarrow$

Apply on Image

Cross Correlation:

$$\begin{bmatrix} k(1,1) & \cdots & k(1,N) \\ \vdots & & \vdots \\ k(N,1) & \cdots & k(N,N) \end{bmatrix}$$

$\downarrow$

Apply on Image.

PSF of Convolution

$$g = \mathcal{O}(f)$$

$$g(\alpha,\beta) = \sum_{i=1}^{N}\sum_{j=1}^{N} f(x,y)\,\underbrace{\left[\mathcal{O}\left(\begin{array}{c}\\ x^{th}\text{ row}\\ \\ \\ \end{array}\left[\begin{array}{ccccc} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{array}\right]\right)\right]_{\alpha,\beta}}_{h^{\alpha,\beta}(x,y)}$$

y^{th} col

PSF is <u>Shift Invariant</u>

if $\exists\, \tilde{h}$,

s.t. $h^{\alpha,\beta}(x,y) = \tilde{h}(\alpha - x, \beta - y)$

$\forall\ \alpha, \beta, x, y$

$$
\boxed{\begin{bmatrix} h^{1,1}(1,1) f_{1,1} & \cdots & h^{1,1}(1,N) f_{1,N} \\ \vdots & & \vdots \\ h^{1,1}(N,1) f_{N,1} & \cdots & h^{1,1}(N,N) f_{N,N} \end{bmatrix}}
\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}
$$

$$
\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}
\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}
$$

$+$

$$
\begin{bmatrix} g(1,1) & g(1,2) & \\ & & \\ & & \end{bmatrix}
$$

$+$

$$
\begin{bmatrix} f_{1,1} & h^{1,2}(1,2) f_{1,2} & \cdots & h^{1,2}(1,N) f_{1,N} \\ \vdots & & & \vdots \\ f_{N,1} & h^{1,2}(N,2) f_{N,2} & \cdots & h^{1,2}(N,N) f_{N,N} \end{bmatrix}
\begin{bmatrix} h^{1,2}(1,1) f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ h^{1,2}(N,1) f_{N,1} & \cdots & f_{N,N} \end{bmatrix}
$$

$$
\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}
\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}
$$

Shift Invariant:

$$\begin{bmatrix} \tilde{h}(0,0) f_{1,1} & \cdots & \tilde{h}(0,1-N) f_{1,N} \\ \vdots & & \vdots \\ \tilde{h}(1-N,0) f_{N,1} & \cdots & \tilde{h}(1-N,1-N) f_{N,N} \end{bmatrix} \begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}$$

$$\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix} \begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}$$

$+$

$$\begin{bmatrix} g(1,1) & g(1,2) & \\ & & \\ & & \end{bmatrix}$$

$+$

$$\begin{bmatrix} f_{1,1} & \tilde{h}(0,0) f_{1,2} & \cdots & \tilde{h}(0,2-N) f_{1,N} \\ \vdots & & & \vdots \\ f_{N,1} & \tilde{h}(1-N,0) f_{N,2} & \cdots & \tilde{h}(1-N,2-N) f_{N,N} \end{bmatrix} \begin{bmatrix} \tilde{h}(0,1) f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ \tilde{h}(1-N,1) f_{N,1} & \cdots & f_{N,N} \end{bmatrix}$$

$$\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix} \begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}$$

Periodically Extend $\tilde{h}$:

$$\begin{bmatrix} \tilde{h}(N,N) f_{1,1} & \cdots & \tilde{h}(N,1) f_{1,N} \\ \vdots & & \vdots \\ \tilde{h}(1,N) f_{N,1} & \cdots & \tilde{h}(1,1) f_{N,N} \end{bmatrix} \begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}$$

$$\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix} \begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}$$

$+$

$$\begin{bmatrix} g(1,1) & g(1,2) & & \\ & & & \\ & & & \end{bmatrix}$$

$+$

$$\begin{bmatrix} f_{1,1} & \tilde{h}(N,N) f_{1,2} & \cdots & \tilde{h}(N,1) f_{1,N} \\ \vdots & & & \vdots \\ f_{N,1} & \tilde{h}(1,N) f_{N,2} & \cdots & \tilde{h}(1,2) f_{N,N} \end{bmatrix} \begin{bmatrix} \tilde{h}(N,1) f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ \tilde{h}(1,1) f_{N,1} & \cdots & f_{N,N} \end{bmatrix}$$

$$\begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix} \begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix}$$

$\therefore$ Shift Invariant $\Leftrightarrow$ Convolution,

$$\tilde{h} = k.$$

<u>Example</u>:

Let O be an Image Transformation, $f \in I$.

and

$$O(f)(\alpha, \beta) = 4 f(\alpha, \beta) - f(\alpha - 1, \beta) - f(\alpha + 1, \beta) - f(\alpha, \beta - 1) - f(\alpha, \beta + 1).$$

Note that O is shift Invariant:

$$\text{Let } P_{xy} = \begin{array}{cc} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & \leftarrow x \\ \uparrow y & \end{array}$$

If $\alpha = x$, $\beta = y$,

$$O(P_{xy})(\alpha, \beta) = 4 \underbrace{f(x, y)}_{1} - \underbrace{f(x-1, y) - \cdots - f(x, y+1)}_{0}$$

$$= 4$$

If $\alpha = x - 1$, $\beta = y$,

$$O(P_{xy})(\alpha, \beta) = 4 f(x-1, y) - f(x-2, y) - f(x, y) - f(x-1, y-1) - f(x-1, y+1)$$

$$= -1$$

Similarly,

$$O(P_{xy})(\alpha,\beta) = \begin{cases} 4, & \text{if } \alpha = x, \beta = y \\ -1, & \text{if } (\alpha = x-1, \beta = y) \\ & \text{or } (\alpha = x+1, \beta = y) \\ & \text{or } (\alpha = x, \beta = y-1) \\ & \text{or } (\alpha = x, \beta = y+1) \\ 0, & \text{else.} \end{cases}$$

$$= \begin{cases} 4 & \text{if } \alpha - x = \beta - y = 0 \\ -1, & \text{if } \alpha - x = -1, \beta - y = 0 \\ -1, & \text{if } \alpha - x = 1, \beta - y = 0 \\ -1, & \text{if } \alpha - x = 0, \beta - y = -1 \\ -1, & \text{if } \alpha - x = 0, \beta - y = 1 \\ 0, & \text{else.} \end{cases}$$

$\therefore$ PSF only depends on $\alpha - x$ and $\beta - y$.

$\therefore$ shift-invariant and hence convolution.

$$k = \tilde{h}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & -1 \\ \vdots & \ddots & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & & 0 & -1 \\ -1 & 0 \cdots 0 & -1 & 4 \end{bmatrix}$$

In some cases,

we may write $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$